

EIGENVALUE ESTIMATES FOR SUBMANIFOLDS OF WARPED PRODUCT SPACES

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ABSTRACT. In this paper, we give lower bounds for the fundamental tone of open sets in minimal submanifolds immersed into warped product spaces of type $N^n \times_f Q^q$, where $f \in C^\infty(N)$. Some applications, also regarding the essential spectrum, illustrate the applicability and the generality of our results.

1. INTRODUCTION

Let M be a connected Riemannian manifold, possibly incomplete, and let $\Delta = \operatorname{div} \circ \nabla$ be the Laplace-Beltrami operator on acting on $C_0^\infty(M)$, the space of smooth functions with compact support. When M is geodesically complete, Δ is essentially self-adjoint, thus there is a unique self-adjoint extension to an unbounded operator, denoted by Δ , whose domain is the set of functions $f \in L^2(M)$ so that $\Delta f \in L^2(M)$, see [17], [19] and [30]. If M is not complete we will always consider the Friedrichs extension of Δ . Denote by $\sigma(-\Delta)$ and $\sigma_{\text{ess}}(-\Delta)$, respectively, the spectrum and the essential spectrum of $-\Delta$. Given an open subset $\Omega \subset M$, the fundamental tone of Ω , $\lambda^*(\Omega)$, is defined by

$$\lambda^*(\Omega) = \inf \sigma(-\Delta) = \inf \left\{ \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2}; f \in H_0^1(\Omega) \setminus \{0\} \right\}.$$

When Ω has compact closure and Lipschitz boundary, $\lambda^*(\Omega)$ coincides with the first eigenvalue $\lambda_1(\Omega)$ of Ω , with Dirichlet boundary data on $\partial\Omega$. Its associated eigenspace is 1-dimensional and spanned by any solution u of

$$\begin{cases} \Delta u + \lambda_1(\Omega)u = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The relations between the fundamental tone of open sets of M and their geometric invariants has been the subject to an intensive research in the past 50 years. Among a huge literature, we limit ourselves to quote the classics [5], [6], [15] and references therein for a detailed picture. In particular, a great effort has been done to estimate the fundamental tone of minimal submanifolds of well-behaved ambient spaces (for instance, in [8], [9], [14], [16] and [18]). In this paper, we move a step further by giving lower bounds for the fundamental tone of manifolds which are minimally immersed in ambient spaces $N^n \times_f Q^q$ carrying a warped product structure, see Theorem 10 below. As we shall see in the last section, the generality of our setting allows applications to submanifolds of cylinders, cones, tubes, improving certain recent results in the literature ([7], [8], [9]). We remark that there have been an increasing interest in the study of minimal and constant mean curvature submanifolds in product spaces $N \times \mathbb{R}$, after the discovery of many beautiful examples such

as those in [24], [25], and this motivates a thorough investigation of the spectrum of such submanifolds. In this respect, we hope that our estimates could be useful.

2. PRELIMINARIES

Isometric immersions

Let M and W be smooth Riemannian manifolds of dimension m and $n + q$ respectively and $\varphi: M \hookrightarrow W$ be an isometric immersion. Consider a smooth function $F: W \rightarrow \mathbb{R}$ and the composition $F \circ \varphi: M \rightarrow \mathbb{R}$. Identifying X with $d\varphi(X)$, the Hessian of $F \circ \varphi$ at $x \in M$ is given by

$$(1) \quad \text{Hess}_M(F \circ \varphi)(x)(X, Y) = \text{Hess}_W F(\varphi(x))(X, Y) + \langle \nabla F, \sigma(X, Y) \rangle_{\varphi(x)},$$

where $\sigma(X, Y)$ is the second fundamental form of φ . Tracing (1) with respect to an orthonormal basis $\{e_1, \dots, e_m\}$,

$$(2) \quad \begin{aligned} \Delta_M(F \circ \varphi)(x) &= \sum_{i=1}^m \left\{ \text{Hess}_W F(\varphi(x))(e_i, e_i) + \langle \nabla F, \sum_{i=1}^m \sigma(e_i, e_i) \rangle \right\} \\ &= \sum_{i=1}^m \text{Hess}_W F(\varphi(x))(e_i, e_i) + m \langle \nabla F, H \rangle, \end{aligned}$$

where $H = m^{-1} \text{tr}(\sigma)$ is the normalized mean curvature vector. Formulae (1) and (2) are well known in the literature, see [22].

Models and Hessian comparisons

Hereafter, we denote with $\mathbb{R}_0^+ = [0, +\infty)$. Let $g \in C^2(\mathbb{R}_0^+)$ be positive in $(0, R_0)$, for some $0 < R_0 \leq \infty$, and satisfying

$$g(0) = 0, \quad g'(0) = 1.$$

The κ -dimensional model manifold \mathbb{Q}_g^κ constructed from the function g is the ball $B_R(o) \subseteq \mathbb{R}^\kappa$ with metric given, in polar geodesic coordinates centered at o , by

$$ds_g^2 = dr^2 + g(r)^2 \langle \cdot, \cdot \rangle_{\mathbb{S}^{\kappa-1}},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{S}^{\kappa-1}}$ is the standard metric on the unit $(\kappa - 1)$ -sphere. The radial sectional curvature and the Hessian of the distance function r on \mathbb{Q}_g^κ are given by the expressions

$$K^{\text{rad}} = -\frac{g''(r)}{g(r)}, \quad \text{Hess } r = \frac{g'(r)}{g(r)} (ds^2 - dr \otimes dr).$$

From the first relation, we see that a model can equivalently be specified by prescribing its radial sectional curvature $G \in C^\infty(\mathbb{R}_0^+)$ and recovering g as the solution of

$$(3) \quad \begin{cases} g'' - Gg = 0, \\ g(0) = 0, \quad g'(0) = 1, \end{cases}$$

on the maximal interval $(0, R_0)$ where $g > 0$.

For the proof of our main results we will make use of the following version of the Hessian Comparison Theorem, see [21] and [27, Chapter 2].

Theorem 1. *Let Q^q be a complete Riemannian q -manifold. Fix a point $o \in Q$, denote by $\rho_Q(x)$ the Riemannian distance function from o and let $D_o = Q \setminus \text{cut}(o)$ be the domain of the normal geodesic coordinates centered at o . Given $G \in C^\infty(\mathbb{R}_0^+)$, let g be the solution of the Cauchy problem (3), and let $(0, R_0) \subseteq [0, +\infty)$ be the maximal interval where g is positive. If the radial sectional curvature of Q satisfies*

$$(4) \quad K_Q^{\text{rad}} \leq -G(\rho_Q) \quad (\text{respectively, } K_Q^{\text{rad}} \geq -G(\rho_Q)),$$

on $B(o, R_0)$, then

$$\text{Hess}_Q \rho_Q \geq \frac{g'(\rho_Q)}{g(\rho_Q)} \left(\langle \cdot, \cdot \rangle_Q - d\rho_Q \otimes d\rho_Q \right) \quad (\text{respectively, } \leq)$$

on $D_o \cap B(o, R_0) \setminus \{o\}$, in the sense of quadratic forms.

Eigenvalues and Eigenfunctions

The generalized version of Barta's Eigenvalue Theorem [4], proved in [9] will be important in the sequel.

Theorem 2. *Let Ω be an open set in a Riemannian manifold M and let $f \in C^2(\Omega)$, $f > 0$ on Ω . Then*

$$(5) \quad \lambda^*(\Omega) \geq \inf_{\Omega} \left(-\frac{\Delta f}{f} \right).$$

We recall that, given a model \mathbb{Q}_g^κ with $g > 0$ on $(0, R_0)$, and given $R \in (0, R_0)$, the first eigenfunction v of the geodesic ball $B_g(R)$ centered at o is radial. This can be easily seen by proving that its spherical mean

$$\bar{v}(r) = \frac{1}{g(r)^{\kappa-1}} \int_{\partial B_g(R)} v$$

is still an eigenfunction associated to $\lambda_1(B_g(R))$ and using the fact that the space of first eigenfunctions has dimension 1. With a slight abuse of notation, we can thus identify the first eigenfunction $v \in C^\infty(B_g(R))$ of $B_g(R)$ with the solution $v: [0, R] \rightarrow \mathbb{R}$ of

$$(6) \quad \begin{cases} v'' + (\kappa - 1) \frac{g'}{g} v' + \lambda_1(B_g(R)) v = 0 & \text{on } (0, R), \\ v(0) = 1, \quad v'(0) = 0, \quad v(R) = 0, \quad v > 0 \text{ on } [0, R]. \end{cases}$$

Note that, multiplying the ODE by $g^{\kappa-1}$, integrating and using the initial condition, one can easily argue that $v' < 0$ on $(0, R]$.

We will need the following technical lemma, which extends a result due to Bessa-Costa, see [7, Lemma 2.4].

Lemma 3. *Let \mathbb{Q}_g^κ be a model manifold with radial sectional curvature $-G(r)$, and suppose that $g' > 0$ on $[0, R]$. Let $v \in C^2(B_g(R))$ be a first positive eigenfunction of $B_g(R) \subset \mathbb{Q}_g^\kappa$. If*

$$(7) \quad \lambda_1(B_g(R)) \geq \kappa \|G_-\|_{L^\infty([0, R])}.$$

Then the following inequality holds:

$$(8) \quad \kappa \frac{g'(t)}{g(t)} v'(t) + \lambda_1(B_g(R)) v(t) \leq 0, \quad t \in (0, R].$$

Proof. For simplicity of notation, we denote by $\lambda = \lambda_1(B_g(R))$. Multiplying (6) by $g^{\kappa-1}$ we deduce that $v(t)$ satisfies the following differential equation:

$$(9) \quad \begin{cases} (g^{\kappa-1}v')' + \lambda g^{\kappa-1}v = 0 & \text{on } (0, R), \\ v(0) = 1, \quad v'(0) = 0, \quad v(R) = 0, \quad v > 0 & \text{on } [0, R). \end{cases}$$

Our aim is to deduce (8) via some modified Sturm-type arguments. In order to do so, we search for a positive function μ solving

$$(10) \quad \kappa \mu'(t) \frac{g'(t)}{g(t)} + \lambda \mu(t) = 0 \quad \text{on } (0, R).$$

Integrating, we get that $\log \mu(t) = -\frac{\lambda}{\kappa} \int_0^t \frac{g(s)}{g'(s)} ds$, thus

$$\mu(t) = e^{\left(-\frac{\lambda}{\kappa} \int_0^t \frac{g(s)}{g'(s)} ds\right)}.$$

The above expression is well defined since $g' > 0$ on $[0, R)$.

Since $\mu'(t) = -\frac{\lambda}{\kappa} \frac{g(t)}{g'(t)} \mu(t)$ we deduce

$$(11) \quad \begin{aligned} \mu'(t)v(t) - v'(t)\mu(t) &= -\frac{\lambda}{\kappa} \frac{g(t)}{g'(t)} e^{\left(-\frac{\lambda}{\kappa} \int_0^t \frac{g(s)}{g'(s)} ds\right)} v(t) - v'(t) e^{\left(-\frac{\lambda}{\kappa} \int_0^t \frac{g(s)}{g'(s)} ds\right)} \\ &= -\frac{1}{\kappa} \frac{g(t)}{g'(t)} e^{\left(-\frac{\lambda}{\kappa} \int_0^t \frac{g(s)}{g'(s)} ds\right)} \left(\kappa \frac{g'(t)}{g(t)} v'(t) + \lambda v(t) \right). \end{aligned}$$

From (11) we see that $\kappa \frac{g'(t)}{g(t)} v'(t) + \lambda v(t) \leq 0$ on $(0, R)$ if and only if

$$\mu'(t)v(t) - v'(t)\mu(t) \geq 0 \quad \text{on } (0, R),$$

and we are going to prove this last inequality.

Differentiating (10) and multiplying by $(1/\kappa)$ both sides of the equality, we have

$$\mu''(t) \frac{g'(t)}{g(t)} + \mu'(t) \left[G(t) - \left(\frac{g'(t)}{g(t)} \right)^2 + \frac{\lambda}{\kappa} \right] = 0,$$

that is,

$$\mu''(t) = -\mu'(t) \frac{g(t)}{g'(t)} \left[G(t) - \left(\frac{g'(t)}{g(t)} \right)^2 + \frac{\lambda}{\kappa} \right].$$

Since $\mu'(t) \frac{g(t)}{g'(t)} = -\frac{\lambda}{\kappa} \mu(t) \left(\frac{g(t)}{g'(t)} \right)^2$ we can rewrite $\mu''(t)$ in the following way:

$$\mu''(t) = \frac{\lambda}{\kappa} \mu(t) \left[G(t) \left(\frac{g(t)}{g'(t)} \right)^2 - 1 + \frac{\lambda}{\kappa} \left(\frac{g(t)}{g'(t)} \right)^2 \right].$$

Multiplying the above equation by $g^{\kappa-1}(t)$, and then adding and subtracting the term $(\kappa-1)g^{\kappa-2}(t)g'(t)\mu'(t)$, we obtain

$$(12) \quad (g^{\kappa-1}\mu')'(t) = -\lambda g^{\kappa-1}(t)\mu(t) \left[-\frac{G(t)}{\kappa} \left(\frac{g(t)}{g'(t)} \right)^2 - \frac{\lambda}{\kappa^2} \left(\frac{g(t)}{g'(t)} \right)^2 + 1 \right].$$

Next, we multiply (12) by $v(t)$ and (9) by $-\mu(t)$, and we add them to get

$$(g^{\kappa-1}\mu')'(t)v(t) - (g^{\kappa-1}v')'(t)\mu(t) = \frac{\lambda}{\kappa} g^{\kappa-1}(t)\mu(t)v(t) \left(\frac{g(t)}{g'(t)} \right)^2 \left[G(t) + \frac{\lambda}{\kappa} \right].$$

Integrating from 0 to t gives

$$(13) \quad g^{\kappa-1}(\mu'v - v'\mu)(t) = \int_0^t \frac{\lambda}{\kappa} g^{\kappa-1}(s) \left(\frac{g(s)}{g'(s)} \right)^2 \left[G(s) + \frac{\lambda}{\kappa} \right] \mu(s)v(s) ds.$$

Now, from (7) we deduce that

$$\frac{\lambda}{\kappa} g^{\kappa-1}(t) \left(\frac{g(t)}{g'(t)} \right)^2 \left[G(t) + \frac{\lambda}{\kappa} \right] \mu(t)v(t) \geq 0,$$

whence $\mu'(t)v(t) - v'(t)\mu(t) \geq 0$ for $t \in (0, R)$, as claimed. \square

Remark 4. It is important to find conditions to ensure (7). For instance, if $-G(r) = B^2$, where B is a positive constant, then the solution g_B of (3) is

$$(14) \quad g_B(r) = B^{-1} \sin(Br), \quad \text{thus} \quad g'_B > 0 \quad \text{on} \quad [0, \pi/(2B)).$$

The function g_B yields the model manifold $\mathbb{Q}_{g_B}^\kappa = \mathbb{S}^\kappa(B^2)$, the κ -dimensional sphere of constant sectional curvature B^2 and diameter $\text{diam}_{\mathbb{S}^\kappa(B^2)} = \pi/B$. Note that the first eigenvalue of the geodesic ball of $\mathbb{S}^\kappa(B^2)$ of radius $R = \pi/2B$ is $\lambda_1(B_{\mathbb{S}^\kappa(B^2)}(\pi/2B)) = \kappa B^2$ and $v(r) = \cos(Br)$ is its first eigenfunction.

When $-G(r) \leq B^2$ and $R \leq \pi/(2B)$, by Sturm's argument a solution g of (3) satisfies

$$\frac{g'}{g} \geq \frac{g'_B}{g_B} > 0 \quad \text{on} \quad \left[0, \frac{\pi}{2B}\right).$$

By Cheng's Comparison Theorem (version proved by Bessa-Montenegro in [10]),

$$\lambda_1(B_g(R)) \geq \lambda_1(B_{g_B}(R)), \quad R \in \left[0, \frac{\pi}{2B}\right).$$

In order to get $\lambda_1(B_g(R)) \geq \kappa \|G_-\|_{L^\infty([0,R])}$ it is sufficient to have

$$(15) \quad \lambda_1(B_{g_B}(R)) = \lambda_1(B_{\mathbb{S}^\kappa(B^2)}(R)) \geq \kappa \|G_-\|_{L^\infty([0,R])}.$$

On the other hand, we can see $\kappa \|G_-\|_{L^\infty([0,R])}$ as a first eigenvalue of a ball of radius \tilde{R} in a κ -dimensional sphere of sectional curvature \tilde{B}^2 , i.e.

$$\kappa \|G_-\|_{L^\infty([0,R])} = \lambda_1(B_{\mathbb{S}^\kappa(\tilde{B}^2)}(\tilde{R})),$$

where $\tilde{R} = \frac{\pi}{2\sqrt{\|G_-\|_{L^\infty([0,R])}}}$ and $\tilde{B}^2 = \|G_-\|_{L^\infty([0,R])}$.

We conclude that the inequality (15) holds, thus $\lambda_1(B_g(R)) \geq \kappa \|G_-\|_{L^\infty([0,R])}$, whenever

$$R \leq \frac{\pi}{2\sqrt{\|G_-\|_{L^\infty([0,R])}}}.$$

Remark 5. We remark that if

$$t \int_t^\infty G_-(s) ds \leq \frac{1}{4} \quad \text{for every } t \in \mathbb{R}^+,$$

where $G_-(s) = \max\{0, -G(s)\}$, both g and g' are strictly positive on \mathbb{R}^+ . This criterion has been proved in [13, Prop. 1.21].

A preliminary computation.

From now on, we will consider the case when the ambient space is a warped product $W^{n+q} = N \times_f Q$ of two Riemannian manifolds $(N^n, \langle \cdot, \cdot \rangle_N)$ and $(Q^q, \langle \cdot, \cdot \rangle_Q)$, with the Riemannian metric on W given by

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_N + f^2 \langle \cdot, \cdot \rangle_Q$$

for some smooth positive function $f: N \rightarrow \mathbb{R}^+$. We fix the index convention

$$1 \leq j, k \leq n, \quad n+1 \leq \alpha, \beta \leq n+q.$$

For $(p, q) \in W$, we choose a chart (U, ψ) on N around p , with coordinate tangent basis $\{\partial_j\} = \{\partial/\partial\psi_j\}$, and a chart (V, ϕ) on Q around q , with basis $\{\partial_\alpha\} = \{\partial/\partial\phi_\alpha\}$. Then, with respect to the product chart $(U \times V, \psi \times \phi)$ around (p, q) , the Hessian of F at (p, q) has components

$$(16) \quad \begin{cases} \text{Hess}_W F(\partial_j, \partial_\kappa) = \text{Hess}_N F(\partial_j, \partial_\kappa), \\ \text{Hess}_W F(\partial_j, \partial_\alpha) = \partial_j \partial_\alpha F - \frac{1}{f} \partial_j f \partial_\alpha F, \\ \text{Hess}_W F(\partial_\alpha, \partial_\beta) = \text{Hess}_Q F(\partial_\alpha, \partial_\beta) + \frac{1}{f} \langle \nabla^N f, \nabla F \rangle_N \langle \partial_\alpha, \partial_\beta \rangle, \end{cases}$$

where $\text{Hess}_N F$ and $\text{Hess}_Q F$ mean respectively $\text{Hess}(F \circ i_N)$ and $\text{Hess}(F \circ i_Q)$ and the inclusions are given by

$$\begin{aligned} i_N: (N, \langle \cdot, \cdot \rangle_N) &\rightarrow N \times_f \{q\} \subseteq N \times_f Q, & x &\mapsto (x, q), \\ i_Q: (Q, \langle \cdot, \cdot \rangle_Q) &\rightarrow \{p\} \times_f Q \subseteq N \times_f Q, & y &\mapsto (p, y). \end{aligned}$$

From (16) we observe that if $F(p, q) = f(p) \cdot h(q)$, where f is the warping function and $h: Q \rightarrow \mathbb{R}$ is a smooth function on Q , then $\text{Hess}_W F$ has a block structure, that is

$$\text{Hess}_W F(X, Z) = 0 \quad \forall X \in T_{(p,q)}(N \times_f \{q\}), Z \in T_{(p,q)}(\{p\} \times_f Q).$$

More precisely, we have the following result.

Lemma 6. *Let $F \in C^\infty(N \times_f Q)$ be given by $F(p, q) = f(p) \cdot h(q)$, where $h \in C^\infty(Q)$. Then*

$$(17) \quad \begin{cases} \text{Hess}_W F(X, Y) = h \text{Hess}_N f(X, Y), \\ \text{Hess}_W F(X, Z) = 0, \\ \text{Hess}_W F(Z, W) = f \text{Hess}_Q h(Z, W) + h \frac{|\nabla^N f|_N^2}{f} \langle Z, W \rangle, \end{cases}$$

for every $X, Y \in T_{(p,q)}(N \times_f \{q\})$ and $Z, W \in T_{(p,q)}(\{p\} \times_f Q)$.

3. MAIN RESULTS

Let $\varphi: M^m \rightarrow N^n \times_f Q^q$, $m > n$, be a minimal immersion. Hereafter, we shall require the following

Assumption 7. Define $\rho_Q(x) = \text{dist}_Q(o, x)$ and suppose that the radial sectional curvature of Q satisfies

$$K_Q^{\text{rad}} \leq -G(\rho_Q), \text{ where } G \in C^\infty(\mathbb{R}_0^+).$$

We assume that the solution g of (3) is positive and $g' > 0$ on $[0, R)$, and that $B_Q(o, R) \subseteq Q \setminus \text{cut}(o)$.

Let $v: \overline{B_g(R)} \rightarrow \mathbb{R}$ be the first eigenfunction of the ball $B_g(R) \subset \mathbb{Q}_g^{m-n}$. As remarked, $v > 0$ on $B_g(R)$, v is radial and (up to normalization) solves

$$(18) \quad \begin{cases} v''(t) + (m - n - 1) \frac{g'(t)}{g(t)} v'(t) + \lambda_1(B_g(R)) v(t) = 0, & t \in (0, R) \\ v(0) = 1, \quad v(R) = 0, \quad v > 0 \text{ on } [0, R), \quad v' < 0 \text{ on } (0, R]. \end{cases}$$

Observe that, when $m = n + 1$, the equation simply becomes

$$v''(t) + \lambda_1(B_g(R)) v(t) = 0.$$

Theorem 8. Let $\varphi: M^m \rightarrow N^n \times_f Q^q$ be an m -dimensional submanifold minimally immersed into $N^n \times_f Q^q$, where Q satisfies Assumption 7 and $m > n$. Suppose that the warping function f satisfies

$$(19) \quad \text{Hess}_N f(\cdot, \cdot) - \frac{|\nabla^N f|_N^2}{f} \langle \cdot, \cdot \rangle_N \leq 0.$$

Let $U \subseteq N$ be an open subset, and let $\Omega \subset \varphi^{-1}(U \times_f B_Q(o, R))$ be a connected component. Then, if R is such that

$$(20) \quad R \leq \frac{\pi}{2\sqrt{\|G_-\|_{L^\infty([0, R])}}}$$

the following estimate holds:

$$(21) \quad \lambda^*(\Omega) \geq \inf_{p \in U} \left(\frac{\lambda_1(B_g(R)) - m |\nabla^N f|_N^2(p)}{|f(p)|^2} \right),$$

where $B_g(R)$ is the geodesic ball of radius R in the model manifold \mathbb{Q}_g^{m-n} or the interval $[-R, R]$ if $m = n + 1$.

Proof. We start defining $F: U \times_f B_Q(o, R) \rightarrow \mathbb{R}$ by $F(p, q) = f(p) \cdot h(q)$, where $h \in C^\infty(B_Q(o, R))$ is given by $h(q) = (v \circ \rho_Q)(q)$ and $v \in C^\infty([0, R])$ is the solution of (18). By Theorem 2, we have that

$$(22) \quad \lambda^*(\Omega) \geq \inf_{\Omega} \left(-\frac{\Delta(F \circ \varphi)}{F \circ \varphi} \right).$$

We are going to give a lower bound for $-\Delta(F \circ \varphi)/(F \circ \varphi)$. Let $x \in \Omega$ and let $\{e_1, \dots, e_m\}$ be an orthonormal basis for $T_x \Omega$. Let $\varphi(x) = (p(x), q(x))$, $t(x) = \rho_Q(q(x))$ and denote by $P_N: T_{(p,q)}(N \times_f Q) \rightarrow T_{(p,q)}(N \times_f \{q\})$ and $P_Q: T_{(p,q)}(N \times_f Q) \rightarrow T_{(p,q)}(\{p\} \times_f Q)$ the orthogonal projections onto the tangent spaces of the

two fibers. Then, by (2) and the minimality of M , the Laplacian of $F \circ \varphi$ at x has the expression

$$\begin{aligned} \Delta(F \circ \varphi)(x) &= \sum_{i=1}^m \text{Hess}_W F(\varphi(x))(e_i, e_i) \\ &= \sum_{i=1}^m \left[\text{Hess}_W F(\varphi(x))(P_N e_i, P_N e_i) + \text{Hess}_W F(\varphi(x))(P_Q e_i, P_Q e_i) \right] \end{aligned}$$

where $W = N \times_f Q$. Using Lemma 6, we deduce and writing $t = t(x)$ for simplicity of notation,

$$\begin{aligned} \Delta(F \circ \varphi)(x) &= v(t) \sum_{i=1}^m \text{Hess}_N f(P_N e_i, P_N e_i)(p) + f(p) \sum_{i=1}^m \text{Hess}_Q v(t)(P_Q e_i, P_Q e_i) \\ (23) \quad &+ v(t) \frac{|\nabla^N f|_N^2}{f}(p) \sum_{i=1}^m \langle\langle P_Q e_i, P_Q e_i \rangle\rangle. \end{aligned}$$

Let $\{E_1, \dots, E_n\}$ be an orthonormal basis for $T_p N$, and consider the tangent basis $\{\partial/\partial\rho_Q, \{\partial/\partial\theta^\gamma\}_{\gamma=n+2}^{n+q}\}$, associated to normal coordinates at Q . Then the set $\{\xi_l\}_{l=1}^{n+q}$ given by

$$\xi_j = E_j \quad \forall j = 1, \dots, n, \quad \xi_{n+1} = \frac{1}{f} \frac{\partial}{\partial\rho_Q}, \quad \xi_\gamma = \frac{1}{f} \frac{\partial}{\partial\theta^\gamma} \quad \forall \gamma = n+2, \dots, n+q$$

is an orthonormal basis of $T_{(p,q)}(N \times_f Q)$. So, we can write e_i as a linear combination of vectors of this basis in the following way:

$$e_i = \sum_{j=1}^n a_i^j \cdot \xi_j + b_i \cdot \xi_{n+1} + \sum_{\gamma=n+2}^{n+q} c_i^\gamma \cdot \xi_\gamma,$$

for constants a_i^j, b_i, c_i^γ satisfying

$$(24) \quad \sum_{j=1}^n (a_i^j)^2 + b_i^2 + \sum_{\gamma=n+2}^{n+q} (c_i^\gamma)^2 = 1, \quad \forall i = 1, \dots, m.$$

From

$$\nabla^Q v(t) = v'(t) \frac{\partial}{\partial\rho_Q}, \quad \text{Hess}_Q v(t) = v'(t) \text{Hess}_Q \rho_Q + v''(t) d\rho_Q \otimes d\rho_Q,$$

we can rewrite (23) in the following way:

$$\begin{aligned} \Delta(F \circ \varphi)(x) &= v(t) \sum_{i=1}^m \text{Hess}_N f(P_N e_i, P_N e_i)(p) + f(p) \sum_{i=1}^m \left[P_Q e_i(v'(t)) \left\langle \frac{\partial}{\partial\rho_Q}, P_Q e_i \right\rangle_Q \right. \\ &\quad \left. + v'(t) \text{Hess}_Q \rho_Q(P_Q e_i, P_Q e_i) \right] + v(t) \frac{|\nabla^N f|_N^2}{f}(p) \sum_{i=1}^m \langle\langle P_Q e_i, P_Q e_i \rangle\rangle \\ &= v(t) \sum_{i=1}^m \left(\text{Hess}_N f(P_N e_i, P_N e_i) + \frac{|\nabla^N f|_N^2}{f} (1 - \langle\langle P_N e_i, P_N e_i \rangle\rangle) \right) (p) \\ &\quad + \frac{1}{f(p)} \left(v''(t) \sum_{i=1}^m b_i^2 + v'(t) \sum_{i=1}^m \sum_{\gamma=n+2}^{n+q} (c_i^\gamma)^2 \text{Hess}_Q \rho_Q \left(\frac{\partial}{\partial\theta^\gamma}, \frac{\partial}{\partial\theta^\gamma} \right) \right). \end{aligned}$$

Using (19) and the fact that v is positive we have

$$\begin{aligned} -\Delta(F \circ \varphi)(x) &\geq -\frac{1}{f(p)} \left[mv(t) |\nabla^N f|_N^2(p) + v''(t) \sum_{i=1}^m b_i^2 \right. \\ &\quad \left. + v'(t) \sum_{i=1}^m \sum_{\gamma=n+2}^{n+q} (c_i^\gamma)^2 \text{Hess}_Q \rho_Q \left(\frac{\partial}{\partial \theta^\gamma}, \frac{\partial}{\partial \theta^\gamma} \right) \right]. \end{aligned}$$

Since $v'(t) \leq 0$, we can apply the Hessian Comparison Theorem, to obtain

$$\begin{aligned} -\Delta(F \circ \varphi)(x) &\geq -\frac{1}{f(p)} \left[mv(t) |\nabla^N f|_N^2(p) + v''(t) \sum_{i=1}^m b_i^2 \right. \\ &\quad \left. + v'(t) \frac{g'(t)}{g(t)} \sum_{i=1}^m \sum_{\gamma=n+2}^{n+q} (c_i^\gamma)^2 \right] \\ &= -\frac{1}{f(p)} \left[v''(t) \sum_{i=1}^m b_i^2 + v'(t) \frac{g'(t)}{g(t)} \left(m - \sum_{i=1}^m \sum_{j=1}^n (a_i^j)^2 - \sum_{i=1}^m b_i^2 \right) \right. \\ &\quad \left. + mv(t) |\nabla^N f|_N^2(p) \right] \end{aligned}$$

where the last equality follows by an algebraic manipulation that uses (24) summed for $i = 1, \dots, m$. Now, by a simple rearranging,

$$\begin{aligned} -\Delta(F \circ \varphi)(x) &\geq -\frac{1}{f(p)} \left[v''(t) + (m - n - 1)v'(t) \frac{g'(t)}{g(t)} - v''(t) \left(1 - \sum_{i=1}^m b_i^2 \right) \right. \\ &\quad \left. + v'(t) \frac{g'(t)}{g(t)} \left(n - \sum_{i=1}^m \sum_{j=1}^n (a_i^j)^2 + 1 - \sum_{i=1}^m b_i^2 \right) + mv(t) |\nabla^N f|_N^2(p) \right]. \end{aligned}$$

From (18) we get

$$\begin{aligned} -\Delta(F \circ \varphi)(x) &\geq \frac{v(t)}{f(p)} \left(\lambda_1(B_g(R)) - m |\nabla^N f|_N^2(p) \right) \\ (25) \quad &+ \frac{1}{f(p)} \left[v''(t) \left(1 - \sum_{i=1}^m b_i^2 \right) - v'(t) \frac{g'(t)}{g(t)} \left(n - \sum_{i=1}^m \sum_{j=1}^n (a_i^j)^2 + 1 - \sum_{i=1}^m b_i^2 \right) \right]. \end{aligned}$$

We claim that the last line of (25) is nonnegative, that is,

$$(26) \quad v''(t) \left(1 - \sum_{i=1}^m b_i^2 \right) - v'(t) \frac{g'(t)}{g(t)} \left(n - \sum_{i=1}^m \sum_{j=1}^n (a_i^j)^2 + 1 - \sum_{i=1}^m b_i^2 \right) \geq 0.$$

To prove this, we substitute $v''(t) = -(m-n-1)v'(t)\frac{g'(t)}{g(t)} - \lambda_1(B_g(R))v(t)$ in (26) to get

$$\begin{aligned}
 (27) \quad & v''(t) \left(1 - \sum_{i=1}^m b_i^2\right) - v'(t) \frac{g'(t)}{g(t)} \left(n - \sum_{i=1}^m \sum_{j=1}^n (a_i^j)^2 + 1 - \sum_{i=1}^m b_i^2\right) = \\
 & - \left((m-n)v'(t) \frac{g'(t)}{g(t)} + \lambda_1(B_g(R))v(t)\right) \left(1 - \sum_{i=1}^m b_i^2\right) \\
 & - v'(t) \frac{g'(t)}{g(t)} \left(n - \sum_{i=1}^m \sum_{j=1}^n (a_i^j)^2\right),
 \end{aligned}$$

so that (26) is equivalent to show that

$$\begin{aligned}
 (28) \quad & - \left((m-n)v'(t) \frac{g'(t)}{g(t)} + \lambda_1(B_g(R))v(t)\right) \left(1 - \sum_{i=1}^m b_i^2\right) \\
 & - v'(t) \frac{g'(t)}{g(t)} \left(n - \sum_{i=1}^m \sum_{j=1}^n (a_i^j)^2\right) \geq 0.
 \end{aligned}$$

Now, in our assumption (20), by Remark 4 it holds

$$\lambda_1(B_g(R)) \geq (m-n)\|G_-\|_{L^\infty([0,R])}.$$

Hence, applying Lemma 3 we infer that

$$(m-n)v'(t) \frac{g'(t)}{g(t)} + \lambda_1(B_g(R))v(t) \leq 0.$$

Moreover, it is clear that $(1 - \sum_{i=1}^m b_i^2) \geq 0$, and finally we observe the inequality

$$\sum_{i=1}^m \sum_{j=1}^n (a_i^j)^2 = \sum_{j=1}^n \left(\sum_{i=1}^m \langle e_i, \xi_j \rangle \right) = \sum_{j=1}^n |P_M \xi_j|^2 \leq \sum_{j=1}^n |\xi_j|^2 = \sum_{j=1}^n 1 = n,$$

where P_M is the projection on M .

Keeping in mind that $v' \leq 0$, this concludes the proof of the claimed (28). From (25) we have

$$(29) \quad - \frac{\Delta(F \circ \varphi)}{F \circ \varphi}(x) \geq \frac{1}{f^2(p)} \left(\lambda_1(B_g(R)) - m |\nabla^N f|_N^2(p) \right).$$

Therefore, by (22) we conclude the desired (21). \square

Remark 9. In the case $N = \mathbb{R}$, we observe that the mean curvature function of the fibers $\{p\} \times_f Q$ is given by $\mathcal{H}(y) = f'(y)/f(y)$. Therefore, condition (19) is equivalent to $f\mathcal{H}' \leq 0$, that is, $\mathcal{H}' \leq 0$. There exists a large class of functions for which $\mathcal{H}' \leq 0$. For instance, $f(y) = \text{constant}$, $f(y) = y$ and $f(y) = e^{cy}$, where $c \in \mathbb{R}$.

4. APPLICATIONS

To show the generality of Theorem 10, we conclude this paper with a number of different examples, and we discuss the sharpness of the estimates produced.

4.1. Cylinders. Considering $f = 1$ and $N = \mathbb{R}$ in Theorem 10 we obtain a generalized version of Theorem 1.1 of [7].

Corollary 10. *Let $\varphi: M^m \rightarrow \mathbb{R} \times Q^q$ be an m -dimensional submanifold minimally immersed into $\mathbb{R} \times Q^q$. Suppose that Q satisfies the Assumption 7. Let $\Omega \subset \varphi^{-1}(\mathbb{R} \times B_Q(o, R))$ be a connected component with*

$$R \leq \frac{\pi}{2\sqrt{\|G_-\|_{L^\infty([0, R])}}}.$$

Then

$$\lambda^*(\Omega) \geq \lambda_1(B_g(R)).$$

Here $B_g(R)$ is a geodesic ball of radius R in an $(m-1)$ -dimensional model manifold \mathbb{Q}_g^{m-1} .

In particular, when $Q^q = \mathbb{R}^q$ in the last corollary we get the following result in the Euclidean space proved by Bessa and Costa in [7].

Corollary 11. *Let $\varphi: M^m \rightarrow \mathbb{R}^{q+1}$ be an m -dimensional submanifold minimally immersed into \mathbb{R}^{q+1} . Let $\Omega \subset \varphi^{-1}(\mathbb{R} \times B_{\mathbb{R}^q}(o, R))$ be a connected component. Then*

$$(30) \quad \lambda^*(\Omega) \geq \lambda_1(B_{\mathbb{R}^{m-1}}(o, R)) = \left(\frac{c_{m-1}}{R}\right)^2.$$

Here c_{m-1} is the first zero of the $J_{(m-1)/2-1}$ -Bessel function.

4.2. Pseudo-hyperbolic and hyperbolic spaces. The pseudo-hyperbolic spaces, introduced by Tashiro in [31], are warped products $\mathbb{R} \times_f Q^q$ with

$$(i) \quad f(y) = ae^{by}, \quad \text{or} \quad (ii) \quad f(y) = a \cosh(by),$$

for some constants $a, b > 0$. In the case (i), we observe that condition (19) is satisfied, as it shows

$$f'' - \frac{(f')^2}{f} = \begin{cases} 0 & \text{in case (i),} \\ ab^2 / \cosh(by) > 0 & \text{in case (ii).} \end{cases}$$

We state the following corollary in the case $f(y) = e^{by}$.

Corollary 12. *Let $\varphi: M^m \rightarrow \mathbb{R} \times_{e^{by}} Q^q$ be an m -dimensional submanifold minimally immersed into $\mathbb{R} \times_{e^{by}} Q^q$. Suppose that Q satisfies Assumption 7. Let $\Omega \subset \varphi^{-1}((\alpha, \beta) \times_{e^{by}} B_Q(o, R))$ be a connected component with*

$$R \leq \frac{\pi}{2\sqrt{\|G_-\|_{L^\infty([0, R])}}}.$$

Then,

$$(31) \quad \lambda^*(\Omega) \geq \frac{\lambda_1(B_g(R))}{e^{2b\beta}} - mb^2.$$

Here $B_g(R)$ is the geodesic ball of $(m-1)$ -dimensional model space \mathbb{Q}_g^{m-1} .

Foliating through horospheres, we can represent the hyperbolic space \mathbb{H}^{q+1} as the warped product $\mathbb{R} \times_{e^y} \mathbb{R}^q$. By Corollary 12 we have the following eigenvalue estimate.

Corollary 13. *Let $\varphi: M^m \rightarrow \mathbb{H}^{q+1}$ be an m -dimensional submanifold minimally immersed into \mathbb{H}^{q+1} . Let $\Omega \subset \varphi^{-1}((-\infty, \beta) \times_{ey} B_{\mathbb{R}^q}(o, R))$ be a connected component. Then*

$$(32) \quad \lambda^*(\Omega) \geq \frac{\lambda_1(B_{\mathbb{R}^{m-1}}(o, R))}{e^{2\beta}} - m = e^{-2\beta} \left(\frac{c_{m-1}}{R} \right)^2 - m,$$

where c_{m-1} is the first zero of the $J_{(m-1)/2-1}$ -Bessel function.

4.3. Cones. A $(q+1)$ -dimensional cone $\mathcal{C}^{q+1}(Q) \subseteq \mathbb{R}^m$ over an open subset $Q \subset \mathbb{S}^q$ can be seen as the warped product $\mathcal{C}^{q+1}(Q) = (0, +\infty) \times_f Q$ where $f(y) = y$. In order to match with Assumption 7 we shall suppose that $Q \subset B_{\mathbb{S}^q}(o, R)$ for some $R \leq \pi/2$. More generally, we can consider cones $\mathcal{C}^{q+1}(Q)$ over open subsets $Q \subset W$ of Riemannian manifolds W with Q satisfying Assumption 7. We have the following result.

Corollary 14. *Let $\varphi: M^m \rightarrow \mathcal{C}^{q+1}(Q)$ be a m -dimensional submanifold minimally immersed into $\mathcal{C}^{q+1}(Q)$ with Q satisfying the Assumption 7. Let $\Omega \subset \varphi^{-1}((0, a) \times_y B_Q(o, R))$ be a connected component with*

$$R \leq \frac{\pi}{2\sqrt{\|G_-\|_{L^\infty([0, R])}}}.$$

Then,

$$(33) \quad \lambda^*(\Omega) \geq \frac{1}{a^2} (\lambda_1(B_g(R)) - m),$$

where $B_g(R)$ is the geodesic ball of radius R in the model manifold \mathbb{Q}_g^{m-1} .

We are ready to analyze the spherical case. Although the sphere is well studied, the values of the first eigenvalue $\lambda_1(B_{\mathbb{S}^m}(r))$ are pretty much unknown, with the exceptions $\lambda_1(B_{\mathbb{S}^m}(\pi/2)) = m$ and $\lambda_1(B_{\mathbb{S}^m}(\pi)) = 0$. We should mention the estimates for spherical cups [1], [28], [29] in dimension two, [20] in dimension three and [2], [3], [12] in all dimensions.

Corollary 15. *Let $\varphi: M^m \rightarrow \mathbb{S}^{q+1} = (o, \pi) \times_{\sin y} \mathbb{S}^q$ be an m -dimensional submanifold minimally immersed into \mathbb{S}^{q+1} . Let $\Omega \subset \varphi^{-1}((o, r) \times_{\sin y} B_{\mathbb{S}^q}(\theta))$, $\theta < \pi/2$ be a connected component. Then*

$$(34) \quad \lambda^*(\Omega) \geq \begin{cases} \frac{\lambda_1(B_{\mathbb{S}^{m-1}}(\theta)) - m}{(\sin r)^2} & \text{if } r \leq \pi/2, \\ \lambda_1(B_{\mathbb{S}^{m-1}}(\theta)) - m & \text{if } r \geq \pi/2. \end{cases}$$

4.4. Essential spectrum. The ideas developed above can be applied to study the essential spectrum of $-\Delta$ of submanifolds properly immersed into the hyperbolic spaces with fairly weak bounds on the mean curvature vector. Via Persson formula ([26] and [11, Prop. 3.2]), one can express the bottom of the essential spectrum of $-\Delta$ as follows: for every exhaustion of M by relatively compact open sets $\{K_j\}$ with Lipschitz boundary,

$$(35) \quad \inf \sigma_{\text{ess}}(-\Delta) = \lim_{j \rightarrow +\infty} \lambda^*(M \setminus K_j).$$

It therefore follows that $-\Delta$ has pure discrete spectrum if and only if

$$\lim_{j \rightarrow +\infty} \lambda^*(M \setminus K_j) = \infty.$$

Our next application regards the essential spectrum of graph hypersurfaces of \mathbb{H}^{q+1} whose boundary lies in a relatively compact region of \mathbb{H}_∞^q , the boundary at infinity of \mathbb{H}^{q+1} .

Corollary 16. *Consider the upper half-space model of the hyperbolic space \mathbb{H}^{q+1} , $q \geq 2$, with coordinates $(x_0, x_1, \dots, x_q) = (x_0, \bar{x})$ and metric*

$$\langle \cdot, \cdot \rangle = \frac{1}{x_0^2} (dx_0^2 + dx_1^2 + \dots + dx_q^2),$$

and let \mathbb{H}_∞^q be its boundary at infinity, with chart \bar{x} . Consider a hypersurface without boundary $\varphi : M^q \rightarrow \mathbb{H}^{q+1}$ that can be written as the graph of a function u over a relatively compact, open set $W \subseteq \mathbb{H}_\infty^q$, and denote with $H(\bar{x})$ its mean curvature. For $z > 0$, define

$$H_z = \sup \{ |H(\bar{x})| : \bar{x} \in W, u(\bar{x}) = z \}$$

If

$$(36) \quad \lim_{z \rightarrow 0} z^2 H_z = 0,$$

then M has pure discrete spectrum.

Proof. Setting $y = \log x_0$, we can rewrite the metric on \mathbb{H}^{q+1} as the one of the warped product $\mathbb{R} \times_{e^y} \mathbb{R}^q$. In our assumptions, since M has no boundary and is a graph over W it holds $y(\varphi(x)) \rightarrow -\infty$ as x diverges in M^q . We identify the factor \mathbb{R}^q in the warped product structure with \mathbb{H}_∞^q endowed with the Euclidean metric, we fix an origin $o \in \mathbb{H}_\infty^q$ and we let R be large enough that $W \subset B_{\mathbb{R}^q}(o, R)$. Let $\{z_j\} \downarrow 0^+$ be a chosen sequence, set $\beta_j = \log z_j \downarrow -\infty$ and define

$$K_j = \varphi^{-1}((\beta_j, +\infty) \times W), \quad \Omega_j = M \setminus K_j.$$

In our assumptions, K_j is relatively compact for every j and $\{K_j\}$ is a smooth exhaustion of M . Consider a positive first eigenfunction v of the geodesic ball $B_{\mathbb{R}^{q-1}}(o, 2R)$, with the normalization $\|v\|_{L^\infty} = 1$. Define $F : (-\infty, \beta_j) \times_{e^y} B_{\mathbb{R}^q}(o, 2R) \rightarrow \mathbb{R}$ as

$$F(y, p) = e^y \cdot h(p),$$

where $h(p) = v(\rho_{\mathbb{R}^q}(p))$. By Theorem 2 and formula (2),

$$\begin{aligned} \lambda^*(M \setminus K_j) &\geq \inf_{M \setminus K_j} \frac{-\Delta(F \circ \varphi)}{F \circ \varphi} \\ &= \inf_{M \setminus K_j} -\frac{1}{F \circ \varphi} \left[\sum_{i=1}^q \text{Hess}_{\mathbb{H}^{q+1}} F(\varphi(x)) (e_i, e_i) + q \langle \nabla F, H \rangle \right]. \end{aligned}$$

The proof of Theorem 10, in particular inequality (29), show that, for $x \in M \setminus K_j$,

$$-\frac{1}{F \circ \varphi} \sum_{i=1}^q \text{Hess}_{\mathbb{H}^{q+1}} F(\varphi(x)) (e_i, e_i) \geq \frac{\lambda_1(B_{\mathbb{R}^{q-1}}(o, 2R))}{e^{2y(x)}} - q,$$

therefore, on $M \setminus K_j$,

$$-\frac{\Delta(F \circ \varphi)}{F \circ \varphi}(x) \geq \frac{\lambda_1(B_{\mathbb{R}^{q-1}}(o, 2R))}{e^{2y(x)}} - q - q |H| \frac{|\nabla F|}{F}(\varphi(x)).$$

On the other hand, $\nabla F = F \nabla y + e^y \nabla h$ and thus $|\nabla F|/F \leq 1 + |\nabla h|/h$. Since $1 \geq h > 0$ on $\overline{B_{\mathbb{R}^{q-1}}(o, R)}$, we infer that

$$\sup_{B_{\mathbb{R}^{q-1}}(o, R)} \frac{|\nabla F|}{F} \leq C(R),$$

where

$$C(R) = 1 + \sup_{B_{\mathbb{R}^{q-1}}(o, R)} \frac{|\nabla h|}{h} > 0.$$

From the above, we have

$$(37) \quad \lambda^*(M \setminus K_j) \geq \inf_{M \setminus K_j} \left[\frac{\lambda_1(B_{\mathbb{R}^{q-1}}(o, 2R)) - qC(R)|H(x)|e^{2y(x)} - qe^{2y(x)}}{e^{2y(x)}} \right].$$

In our assumptions, on $M \setminus K_j$,

$$|H(x)|e^{2y(x)} \leq H_{x_0(x)}e^{2y(x)} = H_{x_0(x)}[x_0(x)]^2.$$

By (36), this latter goes to zero uniformly for $x \in M \setminus K_j$ and divergent j . In particular, for each fixed $\varepsilon > 0$, there exists j_ε large such that, for $j \geq j_\varepsilon$, $|H(x)|e^{2y(x)} \leq \varepsilon$ on $M \setminus K_j$. It therefore follows that, for j large enough,

$$\lambda^*(M \setminus K_j) \geq \inf_{M \setminus K_j} \left[\frac{\lambda_1(B_{\mathbb{R}^{q-1}}(o, 2R)) - qC(R)\varepsilon - qx_0(x)^2}{x_0(x)^2} \right].$$

Choosing ε sufficiently small, letting $j \rightarrow +\infty$ and using that $x_0(x)^2 \leq e^{2\beta_j} \rightarrow 0^+$ for $x \in M \setminus K_j$ and divergent j , we deduce that $\lambda^*(M \setminus K_j) \rightarrow +\infty$, and the claim follows by Persson formula. \square

To conclude, we consider the essential spectrum of submanifolds satisfying some strong non-properness assumption. This includes submanifolds with bounded image immersed in a complete manifold. We begin with recalling the following

Definition 17. *Let M, W be Riemannian manifolds and let $\varphi: M \rightarrow W$ be an isometric immersion. The limit set of φ , denoted by $\lim \varphi$, is a closed set defined as follows*

$$\lim \varphi = \{p \in W; \exists \{p_k\} \subset M, \text{dist}_M(o, p_k) \rightarrow \infty \text{ and } \text{dist}_W(p, \varphi(p_k)) \rightarrow 0\}.$$

Observe that:

- An isometric immersion $\varphi: M \rightarrow W$ is proper if and only if $\lim \varphi = \emptyset$.
- The closure of the set $\varphi^{-1}[W \setminus T_\epsilon(\lim \varphi)]$ may not be a compact subset of M . Here $T_\epsilon(\lim \varphi) = \{y \in W: \text{dist}_W(y, \lim \varphi) < \epsilon\}$ is the ϵ -tubular neighborhood of $\lim \varphi$.

Definition 18. *An isometric immersion $\varphi: M \rightarrow W$ is strongly non-proper if for all $\epsilon > 0$ the closed subset $\varphi^{-1}(W \setminus T_\epsilon \lim \varphi)$ is compact in M .*

Remark 19. A strongly non-proper immersions is not necessarily bounded: for example, the graph immersion $\varphi: B_1(0) \setminus \{0\} \subset \mathbb{R}^m \rightarrow \mathbb{R}^m \times \mathbb{R}$ given by

$$\varphi(x) = (w, z) = \left(x, \frac{1-r(x)}{r(x)} \sin(r(x)(1-r(x))) \right)$$

is strongly non-proper, and $\lim \varphi = \{w = 0\} \cup \{r(w) = 1, z = 0\}$.

Corollary 20. *Let $\varphi: M^m \rightarrow N^n \times_f Q^q$ be a strongly non-proper minimal submanifold. Suppose that Q satisfies Assumption 7. Assume in addition that the warping function f satisfies $\inf_N f > c_1 > 0$, $\sup_N |\nabla f| \leq c_2 < \infty$ and*

$$\text{Hess}_N f(\cdot, \cdot) - \frac{|\nabla^N f|_N^2}{f} \langle \cdot, \cdot \rangle_N \leq 0.$$

Then, if $\lim \varphi \subset N \times_f \{o\}$, the spectrum of M is discrete.

Proof. Let $T_j(N) = N \times_f B_Q(o, 1/j)$, for j large enough that $B_Q(o, 2/j) \Subset M$ is a regular, convex ball. Let $K_j = \varphi^{-1}[(N \times_f Q) \setminus T_j(N)]$ be an exhaustion of M by relatively compact, open sets. Note that $\varphi(M \setminus K_j) \subset T_j(N)$. We now proceed as in the proof of Corollary 16. Define $F = f(p)v_j(\rho_Q(q))$, where v_j is the first eigenfunction of $B_g(2/j) \subset \mathbb{Q}_g^{m-n}$, normalized according to $\|v_j\|_{L^\infty} = 1$, and note that

$$\|\nabla \log F\|_{L^\infty(T_j(N))} \leq \|\nabla \log f\| + \|\nabla \log v_j\| \leq \frac{c_2}{c_1} + \|\nabla \log v_j\|.$$

By gradient estimates (see for instance, [23, Thm. 6.1].)

$$\|\nabla \log v_j\|_{L^\infty(T_j)} = \left\| \frac{v'_j}{v_j} \right\|_{L^\infty([0,j])} \leq C \cdot j,$$

for some absolute constant $C > 0$, and so $\|\nabla \log F\|_{L^\infty(T_j)} \leq Cj$. Using formula (37) and proceeding as in the proof of Corollary 16, we have that

$$\begin{aligned} \lambda^*(M \setminus K_j) &\geq \inf_{p \in N} \left(\frac{\lambda_1(B_g(2/j)) - m|\nabla^N f|_N^2(p)}{|f(p)|^2} \right) \\ &\quad - m\|H\|_{L^\infty(M)} \|\nabla \log F\|_{L^\infty(T_j)}. \end{aligned}$$

Since

$$\frac{\lambda_1(B_g(2/j)) - m|\nabla^N f|_N^2(p)}{|f(p)|^2} \geq \frac{\lambda_1(B_g(2/j)) - mc_2^2}{c_1^2},$$

we deduce

$$\lambda^*(M \setminus K_j) \geq \frac{\lambda_1(B_g(2/j)) - mc_2^2}{c_1^2} - m\|H\|_{L^\infty(M)} Cj.$$

Taking into account the standard asymptotic $\lambda_1(B_g(2/j)) \sim Cj^2$, for some $C > 0$, we conclude that

$$\lim_{j \rightarrow +\infty} \lambda^*(M \setminus K_j) = +\infty,$$

and the thesis follows by Persson formula. \square

Acknowledgements: The first author was partially supported by CNPq, grant # 301041/2009-1. The second author was partially supported by MICINN project MTM2009-10418 and Fundación Séneca project 04540/GERM/06, Spain, by the Inter-university Cooperation Programme Spanish-Brazilian project PHB2010-0137 and by a research training grant within the framework of the programme Research Training in Excellence Groups GERM by Universidad de Murcia. This work was developed during the third author's visiting period at the Universidade Federal do Ceará-UFC, Fortaleza-Brazil. He wishes to thank the Mathematics Department for the warm hospitality and for the delightful research environment. The fourth author was supported by FPI Grant BES-2010-036829 and by was partially supported by

MICINN project MTM2009-10418 and Fundación Séneca project 04540/GERM/06, Spain. This research is a result of the activity developed within the framework of the Programme in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Regional Agency for Science and Technology (Regional Plan for Science and Technology 2007-2010).

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